Chapter 7.

Antidifferentiation.

Situation.

A mathematics teacher taught this mathematics unit to two classes. One Tuesday morning she decided to give one of her classes a test involving ten functions to differentiate. She wrote the functions on the whiteboard and asked her students to write down the derivative of each one. She then went through each one, writing the answers on the board, and the students marked their work.

The teacher had her other class that afternoon and wanted to give them the same ten function test. She went to her classroom early to rub out the answers from the morning lesson. However, when she arrived at the classroom she found that someone had partly rubbed out the questions. What was left is shown below.

Tuesday

Test on differentiation.

f(x)		f '(x)
$5x^2$	1.	$10x$
$x^2 + 4$	2.	$12x$
$\cdot + 3$	3.	7
	4.	4
$+ 3$	5.	$2x + 1$
	6.	x
	7.	$4x + 3$
	8.	$3x^2 - 5$
	9.	$12x^2 + 6x + 2$
	10.	$5x^4 - 1$

- Write a test involving the differentiation of ten functions that would be consistent with the information that was left on the board.
- Is the test you have written necessarily the same as that of others in your class?
- Is the test you have written necessarily the same as the one the teacher gave to her morning class?

Antidifferentiation.

Antidifferentiation is, as its name implies, the opposite of differentiation. If we differentiate x^2 we obtain the derivative, $2x$. If we antidifferentiate $2x$ we return to x^2, an **antiderivative** (or **primitive**) of $2x$. However we do have a problem: There are many functions that differentiate to $2x$.

If $y = x^2$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 + 1$ then $\frac{dy}{dx} = 2x$.

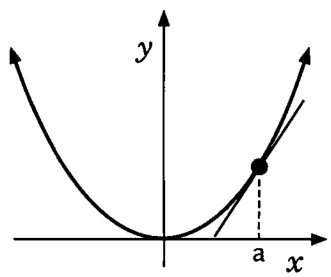
If $y = x^2 - 1$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 + 6$ then $\frac{dy}{dx} = 2x$.

Etc.

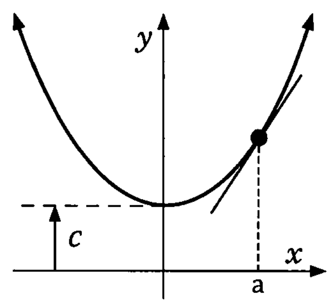
Thus we say that the antiderivative of $2x$ is $x^2 + c$ where c is some constant. Given further information it may be possible to determine the value of this constant, as you will see in example 2.

We can see the need for the "+ c" if we consider the situation graphically.



The diagram on the right shows the graph of $y = x^2$.
The gradient at x = a will be the gradient of the tangent drawn at that point.

If we move $y = x^2$ up by c units the gradient at x = a remains the same.



Thus, if $\frac{dy}{dx} = 2x$, antidifferentiation gives the "family" of curves, $y = x^2 + c$. Each member of this family has the gradient function $2x$ and each member can be obtained from any other by an appropriate vertical shift.

Antidifferentiating powers of x.

In chapter 5 we saw that to differentiate ax^n we used the rule "multiply by the power and decrease the power by one".

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$.

To reverse this process we use the rule:

> "Increase the power by one and divide by the new power."

If $\dfrac{dy}{dx} = anx^{n-1}$ then $y = \dfrac{anx^n}{n}$ ← Power increased by one ← Divide by the new power

$= ax^n$

$$\text{Thus if } \frac{dy}{dx} = ax^n \text{ then } y = \frac{ax^{n+1}}{n+1} + c$$

(Clearly the above rule cannot apply for $n = -1$. Such situations are beyond the scope of this unit.)

Example 1

Find the antiderivative of each of the following.

(a) x^3

(b) $3x^4$

(c) 7

(d) $4x^3 + 12x^2 - 6x$

(a) If $\dfrac{dy}{dx} = x^3$

then $y = \dfrac{x^4}{4} + c$

The antiderivative is $\dfrac{x^4}{4} + c$.

(b) If $\dfrac{dy}{dx} = 3x^4$

then $y = \dfrac{3x^5}{5} + c$

The antiderivative is $\dfrac{3x^5}{5} + c$.

(c) If $\dfrac{dy}{dx} = 7$ (i.e. $7x^0$)

then $y = \dfrac{7x^1}{1} + c$

The antiderivative is $7x + c$.

(d) If $\dfrac{dy}{dx} = 4x^3 + 12x^2 - 6x$

then $y = \dfrac{4x^4}{4} + \dfrac{12x^3}{3} - \dfrac{6x^2}{2} + c$

The antiderivative is $x^4 + 4x^3 - 3x^2 + c$.

The reader should confirm that differentiating each of the above answers does give the required gradient function.

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Example 2

If $\frac{dy}{dx} = 5 - 9x^2$ and, when $x = 1, y = 10$ find (a) y in terms of x,

(b) y, when $x = -1$.

(a) If
$$\frac{dy}{dx} = 5 - 9x^2$$
$$y = 5x - \frac{9x^3}{3} + c$$
$$= 5x - 3x^3 + c$$

We are told that when $x = 1, y = 10$.

Thus $10 = 5(1) - 3(1)^3 + c$

$10 = 5 - 3 + c$ giving $c = 8$.

$\therefore$ $y = 5x - 3x^3 + 8$

(b) If $x = -1$, $y = 5(-1) - 3(-1)^3 + 8$

$= 6$

When $x = -1$, $y = 6$.

Exercise 7A

Find the antiderivative of each of the following.

1.	x^7	2.	x^5	3.	x^4
4.	x^3	5.	x^2	6.	x
7.	1	8.	$12x^2$	9.	$12x^5$
10.	$8x^3$	11.	$14x$	12.	6
13.	$3x^2 + 6x$	14.	$6x^2 - 1$	15.	$7 + 12x^3$
16.	$6x - 15x^4$	17.	$7 - 8x$	18.	$x^2 + 3$
19.	$18x^5 + 1$	20.	$6x^2 + x$	21.	$12x^2 + 8x^3 + 2$
22.	$3x^2 - 2x + x^5$	23.	$1 + x + x^2$	24.	$12x^3 + 6x + 5$

For numbers 25 to 30 expand the given expression and then antidifferentiate.

25. $(3x+4)(x+2)$ 26. $(9x-1)(x+1)$ 27. $(x-2)(x+2)$

28. $(x+1)(x-3)$ 29. $x^2(8x+3)$ 30. $4x(x^2+3x+1)$

31. Find y in terms of x given that $\frac{dy}{dx} = 6x^2$ and $y = 5$ when $x = -1$.

32. Find y in terms of x given that $\frac{dy}{dx} = 3x + 2$ and $y = 0$ when $x = -2$.

33. Find y in terms of x given that $\frac{dy}{dx} = 3x^2 - 2x$ and $y = 6$ when $x = 1$.

34. Find y in terms of x given that $\frac{dy}{dx} = 6x^2 - 5$ and $y = 9$ when $x = 2$.

35. Find y in terms of x given that $\frac{dy}{dx} = 3 + 8x^3$ and $y = 6$ when $x = -1$.

36. If $f'(x) = \frac{3x^2}{2} + 4x - 1$ and $f(-2) = 4$ find (a) $f(x)$,
(b) $f(2)$.

37. If $f'(x) = 3x - 6$ and $f(2) = 0$, find (a) $f(x)$,
(b) $f(-2)$,
(c) a if $f(a) = 54$.

38. A curve has a gradient function of $2x + 7$ and passes through the points (3, p) and (−1, −9). Find the value of p.

39. A curve has a gradient function of $6x^2 - h$ and passes through the origin and the point (4, 0). Find the coordinates of all the points where the curve cuts the x-axis.

40. A curve with a gradient function of $12x - 12$ cuts the x-axis at two points, (3, 0) and (k, 0). Find the value of k.

Function from rate of change.
In the previous chapter, *Applications of differentiation*, we applied our ability to differentiate to real life situations and obtained rates of change such as rate of change of profit, rate of change of area, rate of change of volume, etc. Antidifferentiation can be applied to such rates of change to return to the function for total profit, area, volume, etc.

Example 3
The total revenue raised from the sale of x units of a particular product is $R(x)$ where $R(x)$ is such that:

$$\frac{dR}{dx} = \left(50 - \frac{x}{20}\right) \text{ dollars/unit.}$$

Given that the sale of zero items results in zero revenue find $R(x)$ in terms of x and determine the total revenue resulting from the sale of 100 items.

If $\quad \frac{dR}{dx} = 50 - \frac{x}{20} \quad$ then $\quad R = 50x - \frac{x^2}{40} + c.$

We are told that when $x = 0$, $R = 0$.

i.e. $\quad 0 = 50(0) - \frac{(0)^2}{40} + c \quad$ which gives $c = 0$.

$\therefore \quad R = 50x - \frac{x^2}{40} \quad$ and $R(100) = 4750$.

Thus the total revenue function is $R = 50x - \frac{x^2}{40}$ and the revenue resulting from the sale of 100 items is \$4750.

Exercise 7B

1. If $\frac{dV}{dt} = 6t + 5$ find V in terms of t if $V = 30$ when $t = 0$.

2. If $\frac{dx}{dt} = 2t - 6$, and when $t = 2$, $x = -1$, find
   (a) x in terms of t,
   (b) x, when $t = -2$,
   (c) t, when $x = 2$.

3. If $\frac{dA}{dr} = 4r + 12r^3$, and $A = 7$ when $r = 1$, find
   (a) A in terms of r,
   (b) A, when $r = 2$.

4. For each of the following determine $C(x)$, the total cost function in dollars, from the given information.

   (a) $\frac{dC}{dx} = (2x + 3)$ dollars per unit. $C(0) = 100$.

   (b) $\frac{dC}{dx} = x(3x + 2)$ dollars per unit. $C(0) = 5000$.

5. For each of the following determine $R(x)$, the total revenue function in dollars, given the following information.

   (a) $\frac{dR}{dx} = 50$ dollars per unit. $R(0) = 0$

   (b) $\frac{dR}{dx} = (50 - 0{\cdot}05x)$ dollars per unit. $R(0) = 0$

6. If $R(x)$ is the total revenue from the sale of x items and is such that

$$\frac{dR}{dx} = (400 - 0{\cdot}4x) \text{ dollars per unit}$$

   and $R(0) = 0$, find the total revenue produced from the sale of 100 items.

7. A hole in a balloon causes it to deflate such that the rate of change of volume with respect to time is given by $\frac{dV}{dt} = -(20 + 10t)$ cm^3/sec.

   Find an expression for the volume of the balloon after t seconds given that when $t = 0$, $V = 7000$ cm^3.

8. A, the area of an oil slick, in m^2, t hours after observation commenced was found to be such that $\frac{dA}{dt} = 100$ m^2/hour.

   Is the slick increasing in area or decreasing?

   If the area of the slick was 10 000 m^2 when observation commenced, find a formula for A in terms of t.

9. \$$C$ is the total cost of producing x kg of a particular commodity. The rate of change of C with respect to x is \$40 per kg at all levels of production.

   The fixed costs are \$1000, i.e. $C(0) = 1000$.

   Find C as a function of x.

10. The total revenue for the production and sale of the x units of a particular commodity is given by \$$R(x)$ and is such that: $\frac{dR}{dx} = \frac{2000 - x}{10}$.

    Given that $R(0) = 0$ find $R(x)$ in terms of x and determine $R(1000)$.

11. The manufacturers of a particular sports car find that their sales are usually about 20 cars per week. They run an advertising campaign lasting four weeks. During this time S, the total number of the sports cars sold since the model was first introduced changes such that $\frac{dS}{dt} = 20 + 20t - 3t^2$ where t is the number of weeks the campaign has been running.
Find the number of cars sold (a) in the first week of the campaign,
(b) in the second week of the campaign,
(c) during the four week campaign.

Can a calculator do antidifferentiation for us?

If you continue with later units of *Mathematics Methods* you will see that when we apply a particular "summation" process to the function $f(x) = x^2$, the values we get fit the same function that antidifferentiating x^2 gives, i.e. $\frac{x^3}{3} + c$. This *summation* process is called *integration* and uses the following "stretched S" symbol,

$$\int$$

an s being used due to the link with a **S**ummation process.

Hence with this integration process, that you will meet in a later unit, giving the same function as the antidifferentiation process of this chapter, we tend to use the same stretched S symbol and the word integration when we are determining antiderivatives.

Thus:

- Instead of being asked to find the antiderivative of $6x^2 + 7$ we could be asked to **integrate** $6x^2 + 7$.

- The fact that the antiderivative of $6x^2 + 7$ is $2x^3 + 7x + c$
  could be written as $\int (6x^2 + 7)\,dx = 2x^3 + 7x + c$

- The left hand side of the previous equation tells us to integrate (or antidifferentiate) $6x^2 + 7$, "with respect to x". The stretched S and the dx act like a "wrap" around the expression $6x^2 + 7$. The stretched S telling us that we are antidifferentiating, or integrating, and the dx telling us which variable is involved, in this case, x.
  In this way $\int (6t^2 + 7)\,dt = 2t^3 + 7t + c$

- Using this notation our general rule for antidifferentiating ax^n could be written:

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

- The "+ c", already encountered in this chapter, is called the **constant of integration** (or the constant of antidifferentiation). If sufficient information is given this constant can be determined, as has already been seen.

- Because integrals of the form $\int f(x)\, dx$ involve a constant of integration they are called **indefinite integrals**.

Example 4

Find the following indefinite integrals (a) $\int 12x^2\, dx$, (b) $\int (12x^2 + 2x - 3)\, dx$.

(a) $\int 12x^2 dx = \frac{12x^3}{3} + c$

$= 4x^3 + c$

(b) $\int (12x^2 + 2x - 3)\, dx = \frac{12x^3}{3} + \frac{2x^2}{2} - 3x + c$

$= 4x^3 + x^2 - 3x + c$

So, to answer to the question posed earlier:

Can a calculator do antidifferentiation for us?

some calculators do indeed have the ability to perform the antidifferentiation process but they tend to use the stretched S symbol when determining antiderivatives, as shown on the right.

$\int 6 \cdot x^2 + 7\, dx$ $\quad 2 \cdot x^3 + 7 \cdot x$

$\int 12 \cdot x^2\, dx$ $\quad 4 \cdot x^3$

$\int 12 \cdot x^2 + 2 \cdot x - 3\, dx$ $\quad 4 \cdot x^3 + x^2 - 3 \cdot x$

The displays may feature spaces for entries to be made above and below the integral sign, see right. This is for *definite integrals*, a concept you will meet if you continue to higher units in this course of study. For this unit, if you have access to such a calculator simply leave such entries empty.

$\int_{\square}^{\square} 6 \cdot x^2 + 7\, dx$ $\quad 2 \cdot x^3 + 7 \cdot x$

Note carefully: Calculators tend to omit the "+ c" so we must remember to include it with our answers when determining indefinite integrals.

Exercise 7C.

Find the following indefinite integrals.

1. $\int x^2 \, dx$
2. $\int x \, dx$
3. $\int x^3 \, dx$
4. $\int 2 \, dx$
5. $\int 10x^4 \, dx$
6. $\int 8x^3 \, dx$
7. $\int (4x + 1) \, dx$
8. $\int (6x^2 - 5) \, dx$
9. $\int (8x - 7) \, dx$
10. $\int (x + 9x^2) \, dx$
11. $\int (x - 1) \, dx$
12. $\int (2x + 3)(3x + 1) \, dx$
13. $\int 6x(x + 1) \, dx$
14. $\int x^2(8x - 3) \, dx$
15. $\int 6x(x + 1)^2 \, dx$

Miscellaneous Exercise Seven.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

Express each of the following as a power of ten.

1. 10 000
2. 0·1
3. $10^9 \div 10^3$
4. $10^5 \times 1000$
5. $10^5 \div 1000$
6. $10^7 \times 10^5 \div 1\,000\,000$
7. $\sqrt{10}$
8. $\sqrt[3]{10}$
9. $\sqrt{1\,000}$

Find the value of *n* in each of the following.

10. $5^n = 1$
11. $5^1 = n$
12. $3^n = 243$
13. $3 \times 3^n = 243$
14. $7^3 \times 7^n = 7^8$
15. $10^6 \div 10^n = 100$
16. $10\,000 \times 10^n = 10^9$
17. $2^n \div 4 = 2^6$
18. $9^8 \div (9^4 \times 9^n) = 81$
19. A particular sequence is arithmetic with a common difference of 6 and a second term equal to 16.
    Define the sequence by stating the first term, T_1, and giving T_{n+1} in terms of T_n.
    By how much does the sum of the first fifteen terms of this sequence exceed the fifteenth term?

20. Find $\frac{dy}{dx}$ for each of the following.

(a) $y = 5$ (b) $y = 5x + 5$

(c) $y = 5x^2 + 5x + 5$ (d) $y = 5x^3 + 5x^2 + 5x + 5$

(e) $y = x^2 + \sqrt{x}$ (f) $y = \frac{1}{2x^3}$

21. If $f(x) = 4x^2 - 3x + 2$ find

(a) f(3), (b) f(−1), (c) $f'(x)$, (d) $f'(3)$.

22. Find the gradient of $y = x^2 + 2x + 1$ at the point (3, 16).

23. Find the gradient of $y = 3x^2 - 6x$ at the point (2, 0).

24. Use calculus and algebra to determine the coordinates of any point(s) on the curve $y = x^3 - 21x^2 - 96x + 2900$ where the gradient is zero.

25. Describe the graphs of each of the following in terms of a transformation of the graph of $y = 5^x$.

(a) $y = 5^x + 1$ (b) $y = 5^{x+1}$

(c) $y = 5^{-x}$ (d) $y = \frac{1}{5^x}$

26. Find the average rate of change of the function $y = x^2 + x$ from the point on the curve where $x = 3$ to the point on the curve where $x = 5$.

27. Find the coordinates of the points on the graphs of the given functions where the gradient is as stated.

(a) $y = 3x^2 + 5x$, gradient = −1. (b) $y = x^3 + 2x$, gradient = 5.

28. The quadratic function $y = (x - a)(x + b)$ cuts the x-axis at (−3, 0) and (7, 0). Given that a > 0 and b > 0 determine:

(a) The value of a and of b.

(b) The coordinates of the point where the function cuts the y-axis.

(c) The gradient of the quadratic at the points where it cuts the x-axis.

(d) The coordinates of any points on the quadratic where the gradient of the curve is equal to 6.

(e) The equation of the tangent to the quadratic at the point where the quadratic cuts the y-axis.

29. The curve $y = (x - a)(x + b)$, a > 0 and b > 0, cuts the x-axis at two points P and Q and cuts the y-axis at the point R (0, −12). If point P has coordinates (−4, 0) find a, b and the gradient of the curve at points P, Q and R.

30. The total cost, \$$C$, of producing x units of a particular product is given by

$$C = 20\,000 + 2000x - 20x^2 + \frac{1}{15}x^3.$$

(a) Find an expression for the instantaneous rate of change of C with respect to x.
(b) Find the rate of change of C, with respect to x, when $x = 50$.
(b) If each unit sells for \$2500 find an expression for $P(x)$, the profit made when x units of the product are produced and sold.
(c) Find the rate of change of P, with respect to x, when $x = 50$.

31. What can we conclude from the display on the right about the graph of

$$y = 4x^3 + 9x^2 - 210x + 75 \text{ ?}$$

Define $f(x) = 4x^3+9x^2-210x+75$
Done
solve$\left(\frac{d}{dx}(f(x))=0, x\right)$
$x = -5$ or $x = \frac{7}{2}$
f(−5)
850
f(3·5)
−378.25

32. A rectangular box is to be made to the following requirements:
- The length, l cm, must be twice the width, w cm.
- The 12 edges must have a total length of 6 metres, i.e. $4l + 4w + 4h = 600$, where h cm is the height of the box.

(a) Copy and complete the following table:

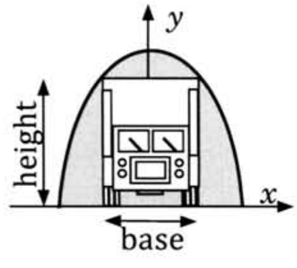
Width (cm)	Length (cm)	Height (cm)	Volume (cm^3)
10			
20			
30			
40			

Continue your table for suitably chosen values for w in order to find, to the nearest centimetre, the dimensions of the box that meet the given requirements and that maximise the volume of the box.

(b) Express the volume of the box in terms of w and use calculus to confirm the answer you obtained in part (a).

33. The diagram on the right shows the cross section of a tunnel with a truck just able to enter.
With units in metres, and x and y axes as shown, the outline of the cross section of the tunnel has equation

$$y = 12 - x^2.$$



Modelling the cross section of this truck as a rectangle, with base and height as indicated, find the dimensions and area of such a rectangle that will just fit into the tunnel, if the area of the rectangle is to be a maximum.